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Asymptotic solutions for linearized disturbances in parallel viscous compressible flows

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Uniformly valid asymptotic solutions for linearized disturbances valid in a finite region containing the critical point $u = c$ are developed using the method of multiple scales for a given isoenergetic basic flow. While uniformity of approximation will certainly improve the quantitative accuracy of a typical computation of stability characteristics of the flow, a major feature of such solutions is that they afford a more accurate description of the behaviour of the linearized disturbances in the neighbourhood of the critical point $u = c$.

1. Introduction

In order to study the behaviour of linearized disturbances in parallel viscous compressible flows one must follow the solution of a linear and homogeneous system of equations which contains time t only through derivatives with respect to t, so that solutions containing an exponential factor e^{ct} may be expected. In the normal-mode theory, each disturbance is resolved into dynamically independent wave components, and each mode of perturbation is assumed to be of the form $\exp[i\alpha(x-ct)]$ times an amplitude function of y, where y is the distance perpendicular to the flow, x the distance along the flow and c the complex wave velocity; $c = c_r + ic_i$, where c_r and c_i are real, $c_i > 0$ implies instability of the perturbed mean flow, $c_i < 0$ implies stability and $c_i = 0$ implies neutral stability. Under the approximations of parallel flow, the equations of continuity, motion, energy and state yield upon reduction five linear differential equations for the amplitudes of the linearized disturbance velocity components, pressure, density and temperature (see Lees & Lin 1946):

$$i(u-c)\tilde{\gamma}+\rho(\phi'+if)+\rho'\phi = 0, \tag{1}$$

$$\alpha\rho[i(u-c)f+u'\phi] = -\frac{i\alpha\pi}{\gamma M_1^2}+\frac{1}{R_L}[f''+\alpha^2(i\phi'-2f)]-\frac{2}{3}\frac{\alpha^2}{R_L}[-f+i\phi'], \tag{2}$$

$$\alpha^2\rho i(u-c)\,\phi = -\frac{\pi'}{\gamma M_1^2}+\frac{\alpha}{R_L}[2\phi''+if'-\alpha^2\phi]-\frac{2}{3}\frac{\alpha}{R_L}[\phi''+if'], \tag{3}$$

$$\alpha\rho[i(u-c)\,\theta+T'\phi] = -\alpha(\gamma-1)\,\rho T(\phi'+if)+\frac{\alpha}{R_L}(\theta''-\alpha^2\theta) + \frac{\gamma(\gamma-1)\,M_1^2}{R_L}2u'(f'+i\alpha^2\phi), \tag{4}$$

$$\frac{\pi}{p} = \frac{\tilde{\gamma}}{\rho} + \frac{\theta}{T}. \tag{5}$$

Here we have considered the simplest model of a compressible fluid, viz. one with a constant ratio of specific heats γ, constant Prandtl number $\mu c_p/k$ (which we take to be unity), constant viscosity coefficient μ and constant thermal conductivity k. In (1)–(5), u is the longitudinal component of the mean velocity, v the transverse component, p the pressure, ρ the mean density and T the mean temperature. The disturbances are defined by

$$\hat{u}, \hat{v}, \hat{p}, \hat{\rho}, \hat{T} = (f, \alpha\phi, \pi, \tilde{\gamma}, \theta) \exp[i\alpha(x - ct)], \tag{6}$$

the non-dimensional parameters being

$$\gamma = \frac{c_p}{c_v}, \quad M_1^2 = \frac{V_\infty^2}{\gamma R T_\infty}, \quad R_L = \frac{\rho_\infty V_\infty L}{\mu}, \tag{7}$$

where the subscript ∞ denotes the conditions in the free stream, R is the gas constant, L a characteristic length and primes denote differentiation with respect to y.

2. Asymptotic solutions

The exact solutions for the disturbances for a general basic compressible flow have not been obtained. The situation of primary interest is when one of the parameters (say R_L) is large, and most of the solutions are constructed within the framework of this feature. Thus the commonly known solutions are valid either in the immediate neighbourhood of the critical point $u = c$ or far away from it.

For $\alpha R_L \gg 1$, the formal asymptotic solutions are (see Morawetz 1954)

$$\phi = \tilde{\phi}^{(0)}(y) + (\alpha R_L)^{-1} \tilde{\phi}^{(1)}(y) + \dots, \tag{8}$$

$$\phi = \exp\left\{-\alpha R_L \left(\int_{y_c}^{y} [i\rho(u-c)]^{\frac{1}{2}} dy\right)\right\} [(u-c)^{-\frac{5}{4}} + \dots], \tag{9}$$

where y_c is the critical point, at which $u(y) = c$. It is known from the inviscid theory that one of the formal asymptotic solutions of the form (8) (denoted by $\tilde{\phi}_1$) is regular, while another (denoted by $\tilde{\phi}_2$) exhibits a logarithmic singularity at the critical point $y = y_c$ (see Shivamoggi (1976*a*) for a comprehensive inviscid theory of the stability of parallel compressible flows). The singularity in this asymptotic solution and the asymptotic solution of the type (9) shows that they can be valid only if the immediate neighbourhood of the critical point is excluded.

All these multi-valued expressions are valid asymptotic solutions in a region which, in the neighbourhood of the point y_c in the complex y plane, is divided into three sectors by the Stokes lines, given by

$$\mathrm{Re}\left\{\int_{y_c}^{y} [i\rho(u-c)]^{\frac{1}{2}} dy\right\} = 0, \tag{10}$$

and it is known that a true solution $\tilde{\phi}_2(y)$ exhibiting the inviscid behaviour $\tilde{\phi}_2^{(0)}(y)$ in two of the sectors will show a dominant viscous behaviour in the remaining sector.

The asymptotic solution valid in the neighbourhood of the critical point is obtained by introducing a new variable (see Lees & Lin 1946)

$$\eta = (y - y_c)/\epsilon, \quad \epsilon = (\alpha R_L)^{-\frac{1}{3}} \ll 1. \tag{11}$$

One then attempts to obtain the solution as a power series in ϵ of the form

$$\phi(\eta;\, \epsilon) = \epsilon\chi_1(\eta) + \epsilon^2\chi_2(\eta) + O(\epsilon^3) \tag{12}$$

which is valid for finite values of η.

The formal identification of (12) with (8) and (9) in the asymptotic limit (which is necessary from a heuristic point of view) raises mathematical difficulties in that it is not known *a priori* whether (8), (9) and (12) have overlapping domains of validity. Besides, in many cases in practice, one boundary corresponds to finite $y - y_c$ while the other corresponds to finite η. Then neither (8), (9) nor (12) is adequate for solving the characteristic-value problem. It is therefore desirable to develop uniformly valid asymptotic solutions for the amplitudes of the linearized disturbances in a finite region containing the critical point, which we accomplish here by using the method of multiple scales for a given basic compressible flow. Two further applications of such uniformly valid asymptotic solutions are (i) in calculating the amplitude distributions for linearized oscillations in parallel compressible flows (which is essential in studying the physical nature of such oscillations), where the commonly known solutions do not provide sufficient accuracy, and (ii) in developing the nonlinear theory for oscillations of finite amplitude, wherein it is essential to have a linear amplitude distribution with a uniformly extended domain of validity. More remarks will be made later, in the discussion.

3. Asymptotic solutions using the method of multiple scales

Consider an isoenergetic basic compressible flow with $M_1 \ll 1$:

$$u = y, \quad T = 1 - \tfrac{1}{2}(\gamma - 1)\, M_1^2 y^2, \quad \rho = 1 + \tfrac{1}{2}(\gamma - 1)\, M_1^2 y^2, \quad p \equiv 1. \tag{13}$$

Introduce two independent variables

$$z = i(y - c), \quad \xi = (\alpha R_L)^{\frac{1}{3}} z \tag{14}$$

(see Cole (1968) for a general outline of the method of multiple scales) and note that

$$\frac{d}{dz} = \frac{\partial}{\partial z} + \frac{1}{\epsilon}\frac{\partial}{\partial \xi},$$

$$\frac{d^2}{dz^2} = \frac{\partial^2}{\partial z^2} + \frac{2}{\epsilon}\frac{\partial^2}{\partial z\, \partial \xi} + \frac{1}{\epsilon^2}\frac{\partial^2}{\partial \xi^2},$$

where
$$\epsilon = (\alpha R_L)^{-\frac{1}{3}} \ll 1,$$
so that (1)–(5) become

$$\begin{aligned} &i\Delta\phi_{,\xi} + \epsilon\{c(\gamma - 1)\, M_1^2(\xi\phi_{,\xi} + \phi) + i\Delta(\phi_{,z} + f)\} \\ &\qquad + \epsilon^2[\xi\tilde{\gamma} + (\gamma - 1)\, M_1^2\{-\tfrac{1}{2}i\xi^2\phi_{,\xi} + c\xi(\phi_{,z} + f) - i\xi\phi\}] + O(\epsilon^3) = 0, \end{aligned} \tag{15}$$

$$[\Delta\phi + i\pi/\gamma M_1^2] + \epsilon[-i(\gamma-1)M_1^2 c\xi\phi + \Delta\xi f + f_{,\xi\xi}] + \epsilon^2[(\gamma-1)M_1^2(-\tfrac{1}{2}\xi^2\phi - ic\xi^2 f) + \tfrac{1}{3}\alpha^2\phi_{,\xi} + 2f_{,\xi z}] + O(\epsilon^3) = 0, \quad (16)$$

$$i\pi_{,\xi}/\gamma M_1^2 + \epsilon[i\pi_{,z}/\gamma M_1^2] + \epsilon^2\alpha^2[\Delta\xi\phi + \tfrac{4}{3}i\phi_{,\xi\xi}] + O(\epsilon^3) = 0, \quad (17)$$

$$i(\gamma-1)\phi' + \epsilon(\gamma-1)[-\Delta M_1^2 c\phi + if] + \epsilon^2[\xi\theta\Delta - i(\gamma-1)M_1^2\xi\phi\Delta + i(\gamma-1)^2 M_1^4 c\xi\alpha\phi + \alpha\theta_{,\xi\xi}] + O(\epsilon^3) = 0, \quad (18)$$

$$[\pi + \delta\gamma + \Delta\theta] + \epsilon i(\gamma-1)M_1^2 c(-\xi\gamma + \xi\theta) + \epsilon^2 \times \tfrac{1}{2}(\gamma-1)M_1^2\xi^2(\gamma+\theta) + O(\epsilon^3) = 0, \quad (19)$$

where $\Delta = 1 + \tfrac{1}{2}(\gamma-1)M_1^2c^2, \quad \delta = 1 - \tfrac{1}{2}(\gamma-1)M_1^2c^2.$

Now seek asymptotic expansions for f, ϕ, π, γ and θ of the form

$$\left.\begin{aligned} f(z,\xi;\epsilon) &= \sum_{n=0}^{\infty}\epsilon^n f_n(z,\xi), \quad \phi(z,\xi;\epsilon) = \sum_{n=0}^{\infty}\epsilon^{n+1}\phi_n(z,\xi),\\ \pi(z,\xi;\epsilon) &= \sum_{n=0}^{\infty}\epsilon^n\pi_n(z,\xi),\\ \gamma(z,\xi;\epsilon) &= \sum_{n=0}^{\infty}\epsilon^n\gamma_n(z,\xi), \quad \theta(z,\xi;\epsilon) = \sum_{n=0}^{\infty}\epsilon^n\theta_n(z,\xi). \end{aligned}\right\} \quad (20)$$

Substituting (20) into (15)–(19) and equating the coefficients of equal powers of ϵ, one obtains

$$\phi_{0,\xi} + f_0 = 0, \quad (21)$$

$$\Delta\phi_0 + \Delta\xi f_0 + f_{0,\xi\xi} = 0, \quad (22)$$

$$\pi_{0,\xi} = 0, \quad (23)$$

$$\xi\theta_0 + \alpha\Delta^{-1}\theta_{0,\xi\xi} - (\gamma-1)M_1^2 c\phi_0 = 0, \quad (24)$$

$$\delta\tilde{\gamma}_0 + \Delta\theta_0 = 0 \quad (25)$$

and

$$i\Delta\phi_{1,\xi} + c\xi(\gamma-1)M_1^2\phi_{0,\xi} + i\Delta\phi_{0,z} + (\gamma-1)M_1^2 c\phi_0 + i\Delta f_1 + \xi\gamma_0 + (\gamma-1)M_1^2 c\xi f_0 = 0, \quad (26)$$

$$\Delta\phi_1 + \Delta\xi f_1 + f_{1,\xi\xi} - i(\gamma-1)M_1^2 c\xi\phi_0 - i(\gamma-1)M_1^2 c\xi^2 f_0 + 2f_{0,\xi z} = 0. \quad (27)$$

From (21)–(27), one finds

$$[\partial^4/\partial\xi^4 + \Delta\xi\,\partial^2/\partial\xi^2]\,\phi_0 = 0, \quad (28)$$

$$[\partial^4/\partial\xi^4 + \Delta\xi\,\partial^2/\partial\xi^2]\,\phi_1 = i(\gamma-1)M_1^2 c[-(1-c)\phi_0 + 2\xi\phi_{0,\xi} + \xi^2\phi_{0,\xi\xi} - \Delta^{-1}\phi_{0,\xi\xi\xi}] - \Delta\phi_{0,z} - \Delta\xi\phi_{0,\xi z} - 3\phi_{0,\xi\xi z} + 2i\xi\tilde{\gamma}_0 + i\xi^2\tilde{\gamma}_{0,\xi} + i\Delta^{-1}\tilde{\gamma}_{0,\xi\xi\xi} + 3\Delta^{-1}\tilde{\gamma}_{0,\xi\xi}. \quad (29)$$

Note that the zeroth-order equation (28) reduces in the limit $M_1 \to 0$ to that deduced by Tam (1968) for the case of an incompressible fluid. The solutions of (28) are integrals of Airy functions. Alternatively, they may be expressed (see Jeffreys 1962, p. 29) as

$$F_k(\zeta) = \int_{L_k} t^{-2}\exp(\zeta t + \tfrac{1}{3}t^3)\,dt, \quad (30)$$

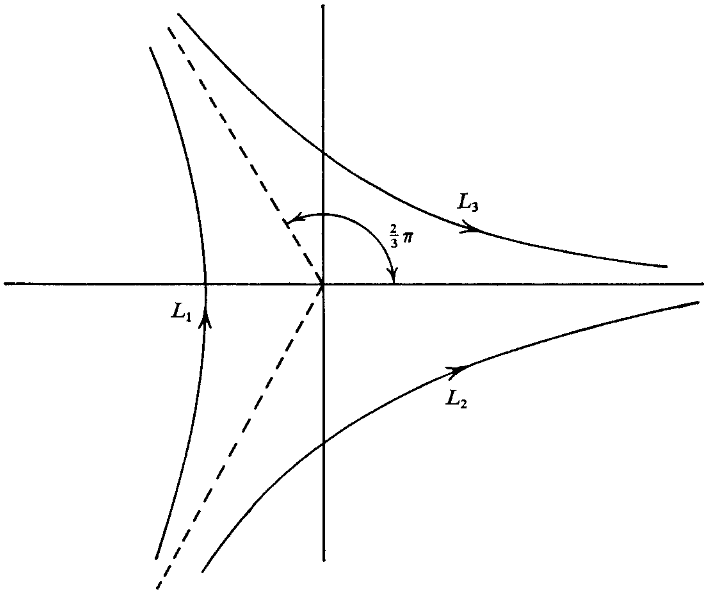

FIGURE 1. Contours for Airy integrals in t plane.

where $\zeta = \Delta^{-\frac{1}{3}}\xi$ and L_k is any one of the contours shown in figure 1. The solutions F_1, F_2 and F_3 are linearly independent; however

$$F_1 + F_2 + F_3 = F_0,$$

where F_0 corresponds to a closed contour L_0 (not shown in the figure) around the origin. Thus one writes

$$\phi_0(z,\zeta) = A_0(z)\,F_1(\zeta) + B_0(z)\,F_2(\zeta) + C_0(z)\,\zeta + D_0(z) \tag{31}$$

if the boundaries where the boundary conditions are to be imposed lie respectively in sectors S_1 and S_2 in figure 2.

It may be seen that the asymptotic solution (31) has coefficients which are functions of the unstretched variable $z = i(y-c)$. In order to determine these functions of z, one imposes the condition that the asymptotic expansion (20) be uniformly valid, i.e. the terms of the asymptotic expansion (20) must satisfy

$$\epsilon\phi_n(z,\xi) = o(\phi_{n-1}(z,\xi))$$

uniformly in z, as $\epsilon \to 0$, the same being true for their partial derivatives. This is accomplished by removing the secular terms in the next higher-order equation, viz. (29).

Inspection of (29) using (31) shows that in order to remove the secular terms one requires

$$\left.\begin{aligned} A_0'(z) = B_0'(z) = C_0'(z) &= 0, \\ D_0'(z) + i(\gamma-1)\,M_1^2 c(1-c)\,\Delta^{-1} D_0(z) &= 0, \end{aligned}\right\} \tag{32}$$

or

$$\left.\begin{aligned} A_0(z) = A_0, \quad B_0(z) = B_0, \quad C_0(z) &= C_0, \\ D_0(z) = \hat{D}_0 + D_0 \exp\left[-i(\gamma-1)\,M_1^2 c(1-c)\,\Delta^{-1} z\right]&. \end{aligned}\right\} \tag{33}$$

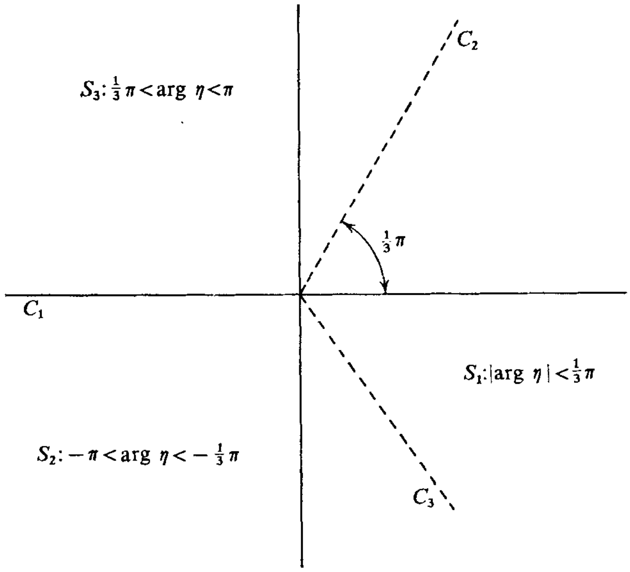

FIGURE 2. Asymptotic domains of Airy integrals in η plane.

Thus one finds a uniformly valid solution

$$\phi_0(z,\xi) = A_0F_1(\zeta) + B_0F_2(\zeta) + C_0\zeta + \hat{D}_0 + D_0 \exp\left[-i(\gamma-1)M_1^2 c(1-c)\Delta^{-1}z\right]. \qquad (34)$$

The asymptotic forms of $F_k(\zeta)$ have been given by Tam:

$$F_k(\zeta) \sim \sim i\pi^{\frac{1}{2}}(-1)^k \exp\left(-i\tfrac{2}{3}\zeta^{\frac{3}{2}}\right)\left(i\tfrac{2}{3}\zeta^{\frac{3}{2}}\right)^{-\frac{1}{2}}\left(i\zeta^{\frac{3}{2}}\right)^{\frac{1}{3}(\lambda+1)} \times \left[1+O\left(|\zeta|^{-\frac{3}{2}}\right)\right] \quad \text{as} \quad |\zeta| \to \infty, \quad \lambda = -2. \qquad (35)$$

Tam further deduced that in each of the sectors S_k, $k = 1, 2, 3$, $F_k(\zeta)$ is dominated by the other two functions.

4. Discussion

While the lack of uniformity of approximation in the conventional asymptotic solutions (8), (9) and (12) will certainly be rectified by any kind of uniformly valid solution, the extent of the usefulness of such a solution will still be determined by the convenience of its form. This would not be a serious issue if one were merely confronted with the computation of the stability characteristics of the flow. However, the primary purpose of the theory of hydrodynamic stability is to focus attention on the mechanism of transition of laminar flow into turbulence. In this context a detailed knowledge of the behaviour of the linearized disturbances in the neighbourhood of the critical point $u = c$ (which in the viscous case can be shown to be a turning point of the mathematical problem) has considerable heuristic value. This also has other important applications: this author (1976*b*) has investigated the influence of 'subsonic' disturbances present in a high-speed gas stream upon the development of the surface waves generated in a liquid layer adjacent to the gas stream. It was found in this case that energy is transferred from the gas stream to the surface waves predominantly by the Fourier component of the gas pressure field in phase with the wavy interface, and the calculation of this energy transfer is facilitated by noting that it is primarily determined by the conditions at the critical point $u = c$.

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